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Spectral Determinants for Schrödinger Equation and Q-Operators of Conformal Field Theory

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Relation between the vacuum eigenvalues of CFT **Q**-operators and spectral determinants of one-dimensional Schrödinger operator with homogeneous potential, recently conjectured by Dorey and Tateo for special value of Virasoro vacuum parameter p, is proven to hold, with suitable modification of the Schrödinger operator, for all values of p.

KEY WORDS: Schrödinger equation; **Q**-operators; vacuum eigenvalues.

In recent remarkable paper[1] a novel relation was observed between the vacuum eigenvalues of so-called **Q**-operators introduced in ref. 2 and the spectral characteristics of the Schrödinger equation

$$\{-\partial_x^2 + |x|^{2\alpha}\}\ \Psi(x) = E\Psi(x) \tag{1}$$

Namely, for special value of the vacuum parameter p (see below) the vacuum eigenvalues of the operators $\mathbf{Q}_+$ and $\mathbf{Q}_-$ essentially coincide with the spectral determinants associated with the odd and even sectors of (1), respectively. In this note we show that a similar relation, with an appropriately generalized spectral problem (1), holds for all values of the vacuum parameter p.

The **Q** operators were constructed in ref. 2 in our attempt to understand $c<1$ CFT as completely integrable theory; they appear to be the CFT versions of Baxter's Q-matrix which plays most important role in his

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famous solution of the eight-vertex model.[3] The **Q**-operators of ref. 2 are actually the operator functions $\mathbf{Q}_{\pm}(\lambda)$, where λ is a complex parameter. These operators act in Virasoro module with the highest weight Δ, this weight and the Virasoro central charge c being conveniently parameterized as $\Delta=(p/\beta)^2+(c-1)/24$ and $c=1-6(\beta-\beta^{-1})^2$, with $0<\beta^2\leqslant 1$. The highest weight state $|p\rangle$ (the Virasoro vacuum) is an eigenstate of $\mathbf{Q}_{\pm}(\lambda)$ and we use the notation

$$\lambda^{\pm 2\pi i p/\beta^2} A_{\pm}(\lambda, p)=\langle p|\, \mathbf{Q}_{\pm}(\lambda)\, |p\rangle \tag{2}$$

for the corresponding eigenvalues. These eigenvalues deserve most detailed study, for various reasons (see refs. 2, 4). Let us mention here some of their properties relevant to the present discussion (details and derivations can be found in refs. 2, 4).

(i) $A_{\pm}(\lambda, p)$ are entire functions of the variable λ^2 with known asymptotic behavior

$$\log A_{\pm}(\lambda, p)\simeq M(-\lambda^2)^{1/(2-2\beta^2)}, \qquad |\lambda^2|\to\infty, \qquad \arg(-\lambda^2)<\pi \tag{3}$$

where

$$M=\frac{\Gamma(\beta^2/(2-2\beta^2))\, \Gamma((1-2\beta^2)/(2-2\beta^2))}{\sqrt{\pi}}\,(\Gamma(1-\beta^2))^{1/(1-\beta^2)}$$

(ii) $A_{+}(\lambda, p)$ is meromorphic function of p, analytic in the half-plane $\Re e(2p)>-\beta^2$, and $A_{-}(\lambda, p)=A_{+}(\lambda, -p)$. The coefficients $a_n(p)$ of the power series expansion

$$\log A_{+}(\lambda, p)=-\sum_{n=1}^{\infty} a_n(p)\,\lambda^{2n} \tag{4}$$

exhibit the following asymptotic behavior

$$a_n(p)\sim p^{1-2n+2n\beta^2} \qquad \text{as} \quad p\to\infty \tag{5}$$

in the half-plane $\Re e(2p)>-\beta^2$.

(iii) $A_{\pm}(\lambda, p)$ satisfy the functional relation (so-called quantum Wronskian condition)

$$\begin{aligned} &e^{2\pi i p} A_{+}(q^{1/2}\lambda, p)\, A_{-}(q^{-1/2}\lambda, p)-e^{-2\pi i p} A_{+}(q^{-1/2}\lambda, p)\, A_{-}(q^{1/2}\lambda, p)\\ &\quad =2i\sin(2\pi p) \end{aligned} \tag{6}$$

where

$$q = e^{i\pi\beta^2} \tag{7}$$

It is important to stress here that above conditions (i)–(iii) define the functions $A_+(\lambda, p)$ uniquely (see Appendix in ref. 4).

Remarkably, according to ref. 1, in the special case $p = \beta^2/4$ the zeroes λ_n^2 of $A_+(\lambda, \beta^2/4)$ and $A_-(\lambda, \beta^2/4)$ coincide, up to an overall factor, with the energy eigenvalues E_n of the Schrödinger equation (1) with

$$\alpha = \frac{1}{\beta^2} - 1 \tag{8}$$

More precisely, let E_n^- and E_n^+ $(n = 1, 2, 3,...)$ be ordered eigenvalues corresponding to even and odd eigenfunctions in (1), respectively, and define the spectral determinants,[3]

$$D^{\pm}(E) = \prod_{n=1}^{\infty} \left(1 - \frac{E}{E_n^{\pm}}\right) \tag{9}$$

The main statement of ref. 1 is

$$A_{\pm}(\lambda, \beta^2/4) = D^{\pm}(\rho\lambda^2) \tag{10}$$

where

$$\rho = (2/\beta^2)^{2-2\beta^2}\, \Gamma^2(1-\beta^2) \tag{11}$$

The key part of the arguments in ref. 1 leading to (10) is the functional relation which the spectral determinants (9) obey; this relation was previously found in important series of works;[(5–7)] it turns out to be identical to the functional relation (6).

We are going to show that a relation similar to (10) holds for generic values of p, if one replaces (1) by the following more general spectral problem. Consider the Schrödinger equation

$$\partial_x^2 \Psi(x) + \left\{E - x^{2\alpha} - \frac{l(l+1)}{x^2}\right\} \Psi(x) = 0 \tag{12}$$

[3] Our notations for these functions slightly differ from those in ref. 1—the superscripts $\pm$ are interchanged and the sign of the argument E is reversed. In addition, there is a difference in normalization—ours corresponds to $D^{\pm}(0) = 1$.

on the half-line $0 < x < \infty$. Here

$$l = \frac{2p}{\beta^2} - \frac{1}{2} \tag{13}$$

and again α is related to β^2 as in Eq. (8). Let us assume that $\Re e\, l > -\frac{3}{2}$, and denote by $\psi(x, E, l)$ the solution of (12) uniquely specified by the condition

$$\psi(x, E, l)\colon \psi(x, E, l) \to \sqrt{\frac{2\pi}{1+\alpha}}\,(2+2\alpha)^{-(2l+1)/(2+2\alpha)} \times \frac{x^{l+1}}{\Gamma(1+((2l+1)/(2+2\alpha)))} + O(x^{l+3}) \qquad \text{as} \quad x \to 0 \tag{14}$$

This solution can be analytically continued outside the domain $\Re e\, l > -\frac{3}{2}$. Obviously thus defined function $\psi(x, E, -l-1)$ solves the same equation (12), and for generic values of l the solutions

$$\psi^+(x, E) = \psi(x, E, l), \qquad \psi^-(x, E) = \psi^+(x, E, -l-1) \tag{15}$$

are linearly independent, since

$$W[\psi^+, \psi^-] = 2i(q^{l+1/2} - q^{-l-1/2}) \tag{16}$$

where $W[f, g] = f\,\partial_x g - g\,\partial_x f$ is the usual Wronskian. For certain isolated values of E one of these solutions decays at $x \to +\infty$. Let $\{E_n^+\}_{n=1}^{\infty}$ and $\{E_n^-\}_{n=1}^{\infty}$ be ordered spectral sets defined by the conditions

$$\begin{aligned} \psi_n^+(x) &\equiv \psi^+(x, E_n^+) \to 0 \\ \psi_n^-(x) &\equiv \psi^-(x, E_n^-) \to 0 \end{aligned} \tag{17}$$

as $x \to +\infty$, and let

$$D^{\pm}(E, l) = \prod_{n=1}^{\infty} \left(1 - \frac{E}{E_n^{\pm}}\right) \tag{18}$$

Simple WKB analysis of the equation (12) shows that

$$E_n^{\pm} \sim n^{2\alpha/(1+\alpha)} \qquad \text{as} \quad n \to \infty \tag{19}$$

and therefore for $\alpha > 1$ these products converge, and (18) defines entire functions of E. It is easy to see that in the special case $l = 0$ the sets

$\{E_n^+\}_{n=1}^{\infty}$ and $\{E_n^-\}_{n=1}^{\infty}$ become the components of the spectrum of (1) associated with odd and even sectors, respectively, and so for $l=0$ the functions (18) reduce to (9). In what follows we will show that for $\alpha>1$ and all values of p

$$A_{\pm}(\lambda, p)=D^{\pm}(\rho\lambda^2, 2p/\beta^2-1/2) \tag{20}$$

We start with an observation that the following transformations of the variables (x, E, l),

$$\hat{\Lambda}\colon x\to x, \qquad E\to E, \qquad l\to -1-l \tag{21}$$

$$\hat{\Omega}\colon x\to qx, \qquad E\to q^{-2}E, \qquad l\to l \tag{22}$$

with $q=e^{i\pi/(1+\alpha)}$, leave the equation (12) unchanged while acting non-trivially on its solutions. As usual, the equation (12) admits a unique solution which decays at large x; we denote this solution as $\chi(x, E, l)$ and fix its normalization by the condition

$$\chi(x, E, l)\colon \chi(x, E, l)\to x^{-\alpha/2}\exp\left\{-\frac{x^{1+\alpha}}{1+\alpha}+O(x^{1-\alpha})\right\} \qquad \text{as} \quad x\to+\infty \tag{23}$$

The transformation $\hat{\Omega}$ applied to $\chi(x, E, l)$ yields another solution, and the pair of functions

$$\chi^+(x, E)=\chi(x, E, l), \qquad \chi^-(x, E)=iq^{-1/2}\chi(qx, q^{-2}E, l) \tag{24}$$

form a basis in the space of solutions of (12). It is not difficult to check that

$$W[\chi^+, \chi^-]=2 \tag{25}$$

i.e., the solutions (24) are indeed linearly independent. The solutions (15) can always be expanded in this basis, in particular

$$\psi^+=C(E, l)\,\chi^+ + D(E, l)\,\chi^- \tag{26}$$

with some nonsingular coefficients $C(E, l)$ and $D(E, l)$. The transformations (21) and (22) act on the solutions (15) and (24) as follows,

$$\hat{\Lambda}\psi^{\pm}=\psi^{\mp}; \qquad \hat{\Lambda}\chi^{\pm}=\chi^{\pm} \tag{27}$$

$$\hat{\Omega}\psi^{\pm}=q^{1/2\pm l\pm 1/2}\psi^{\pm}; \qquad \hat{\Omega}\chi^+=-iq^{1/2}\chi^-, \qquad \hat{\Omega}\chi^-=iq^{1/2}\chi^+ + u\chi^- \tag{28}$$

with some coefficient $u = u(E, l)$. It follows from (28) that

$$C(E, l) = -iq^{-l-1/2}\, D(q^{-2}E, l) \tag{29}$$

Also, applying (27) to (26) one obtains

$$\psi^{-} = D(E, -l-1)\,\chi^{-} - iq^{l+1/2}\, D(q^{-2}E, -l-1)\,\chi^{+} \tag{30}$$

Let us mention here a useful identity

$$D(E, l) = \tfrac{1}{2} W[\chi^{+}, \psi^{+}] \tag{31}$$

At this point we are ready to prove that the pair of functions $D(E, l)$ and $D(E, -1-l)$ satisfy all the conditions (i–iii) above and, since these conditions characterize these functions uniquely,

$$D(\rho\lambda^2, \pm 2p/\beta^2 - 1/2) = A_{\pm}(\lambda, p) \tag{32}$$

Indeed, the analyticity conditions in (i) and (ii) can be derived from (31), while asymptotics there are established by a straightforward WKB analysis of the equation (12). Finally, combining Eqs. (26), (29), (30), (16), (25) one obtains the relation

$$\begin{aligned} q^{l+1/2}\, D(q^2E, l)\, D(E, -l-1) - q^{-l-1/2}\, D(E, l)\, D(q^2E, -l-1) \\ = q^{l+1/2} - q^{-l-1/2} \end{aligned} \tag{33}$$

which is identical to (6).

To prove our statement (20) it remains to show that the coefficient $D(E, l)$ in (26) coincides with the function $D^{+}(E, l)$ defined in (18). Both are entire functions of E. As follows from (26), these functions share the same set of zeroes in the variable E and hence $F(E, l) = \log(D^{+}(E, l)/D(E, l))$ is an entire function of E. However, $E \to \infty$ asymptotic form of $D^{+}(E, l)$ is controlled by asymptotic $n \to \infty$ density of the eigenvalues E_n^{+} which can be computed semiclassically. The result shows that $F(E, l) \to 0$ as $E \to \infty$ and hence $F(E, l) = 0$. Although strictly speaking our proof of (20) is valid only if $\alpha > 1$, the above arguments and the definition (18) can be modified to accommodate wider range of this parameter. We will not elaborate this point here.

In a few special cases the function $A_{+}(\lambda, p)$ was calculated explicitly.[2, 8] Let us consider these examples to illustrate the identity (20). First, for harmonic oscillator case $\alpha = 1$ (which corresponds the $c = -2$ CFT, i.e., the "free fermion" theory) the spectrum of (12) is very well known

$$E_n^+ = 4n + 2l - 1, \qquad n = 1, 2... \tag{34}$$

which allows one to obtain

$$D^+(E, l)|_{\alpha=1} = \frac{\Gamma(3/4 + l/2)\, e^{\mathscr{C}E}}{\Gamma(3/4 + l/2 - E/4)} \tag{35}$$

where $\mathscr{C}$ is a constant whose value depends on the choice of Weierstrass factors required in this case to make the product (18) convergent. The Eq. (35) is identical to the known expression for $A_+(\lambda, p)|_{\beta^2=1/2}$.[2] Next, in the limit $\alpha \to +\infty$ and l fixed (which corresponds to the classical limit $c \to -\infty$ in CFT) the equation (12) reduces to the radial Schrödinger equation for the spherically symmetric "rigid well" potential

$$x^{2\alpha}|_{\alpha \to +\infty} = \begin{cases} 0, & \text{if } \quad 0 < x < 1 \\ +\infty, & \text{if } \quad x > 1 \end{cases} \tag{36}$$

for which the energy levels are related to the zeroes of the Bessel function, and (18) yields

$$D^+(E, l)|_{\alpha \to +\infty} = \Gamma(l + 3/2)(\sqrt{E}/2)^{-l-1/2}\, J_{l+1/2}(\sqrt{E}) \tag{37}$$

where $J_\nu(z)$ is the conventional Bessel function. Again, this expression coincides with the limiting form of $A_+(\lambda, p)|_{\beta^2 \to 0}$.[2] Finally, for $\alpha = 1/2$ and $l = 0$ the spectral determinants (18) are expressed in terms of the Airy function,[7, 1] in agreement with $A_\pm(\lambda, 1/6)|_{\beta^2=2/3}$ obtained in ref. 8.

We would like to mention also some possible applications of the relation (20). In ref. 2 the exact asymptotic expansions for $A_\pm(\lambda, p)$ at large λ were found. The coefficients in these expansions are expressed in terms of the spectral characteristics of CFT, namely the vacuum eigenvalues of its local and non-local integrals of motion. In view of (20) these expansions can be used to derive the large n asymptotics of the energy levels $E_n^\pm$. In particular, the leading terms read

$$\begin{aligned} E_n^+ = {} & \left[\frac{\sqrt{\pi}\,\Gamma(3/2 + 1/2\alpha)}{2\Gamma(1 + 1/2\alpha)}\right]^{2\alpha/(1+\alpha)} (4n + 2l - 1)^{2\alpha/(1+\alpha)} \\ & \times \left(1 - \frac{2\alpha \cot(\pi/2\alpha)}{3\pi(1+\alpha)^2}\, \frac{12l^2 + 12l - 2\alpha + 1}{(4n + 2l - 1)^2} + O(n^{-4}, n^{-1-2\alpha})\right) \quad \left(\alpha > \frac{1}{2}\right) \end{aligned}$$

$$E_n^+ = \left[\frac{\sqrt{\pi}\,\Gamma(3/2+1/2\alpha)}{2\Gamma(1+1/2\alpha)} \right]^{2\alpha/(1+\alpha)} (4n+2l-1)^{2\alpha/(1+\alpha)}$$

$$\times \left(1 + \frac{2\alpha\Gamma(-1/2-\alpha)}{(1+\alpha)\sqrt{\pi}\,\Gamma(-\alpha)} \left[\frac{2\Gamma(1+1/2\alpha)}{\sqrt{\pi}\,\Gamma(3/2+1/2\alpha)} \right]^{2\alpha} \right.$$

$$\times \frac{\Gamma(l+\alpha+3/2)}{\Gamma(l-\alpha+1/2)} (4n+2l-1)^{-2\alpha-1}$$

$$\left. + O(n^{-2}, n^{-1-4\alpha}) \right) \quad \left(0 < \alpha < \frac{1}{2} \right)$$

$$E_n^+ = (3\pi(4n+2l-1))^{2/3} \left(\frac{1}{4} + \frac{12l(l+1)\log(3\pi(4n+2l-1))}{27\pi^2(4n+2l-1)^2} \right.$$

$$\left. - \frac{18l^2+6l-5+12l(l+1)\,\psi(l+2)}{27\pi^2(4n+2l-1)^2} + O(n^{-3}) \right) \quad \left(\alpha = \frac{1}{2} \right) \tag{38}$$

where $\psi(z) = \partial_z \log \Gamma(z)$ and $n \to \infty$. Using expressions for the local and non-local integrals of motion given in refs. 9, 4, it is not difficult to extend these expansions substantially further, but first few terms already yield remarkable accuracy even for lower levels $n \geqslant 4$.

The vacuum eigenvalues of the **Q**-operators, i.e., the above functions $A_\pm(\lambda, p)$, appear also in studying a quantum problem of a Brownian particle in the potential $U(X) = -2\pi\kappa\cos(X) - VX$ at finite temperature T. In the Caldeira–Leggett approach[10] this problem is related to so-called boundary sine-Gordon model (with zero bulk mass),[11] and it was shown in ref. 4 that some expectation values in this problem are expressed through the above functions $A_\pm(\lambda, p)$ with pure imaginary $p \propto iV$ and $\lambda \propto i\kappa$. In particular, the nonlinear mobility $J(V) = \langle \dot{X} \rangle$ is

$$J(V) = V + i\pi T\kappa\partial_\kappa \log \left[\frac{A_+(\sigma\kappa,\ -i\,(V\beta^2/4\pi T))}{A_-(\sigma\kappa,\ -i\,(V\beta^2/4\pi T))} \right] \tag{39}$$

where $\beta^2 \propto \hbar$ is a quantum parameter and $\sigma = i(\beta^2/2\pi T)^{1-\beta^2} \sin(\pi\beta^2)/\beta^2$ (see ref. 4 for details). The equation (39) has nice interpretation in terms of the Schrödinger problem (12). To see this let us make a variable transformation in (12), $x = e^y$, $\Psi = e^{y/2}\tilde{\Psi}$, which brings (12) to the form

$$-\partial_y^2 \tilde{\Psi} + \{ e^{2y/\beta^2} - Ee^{2y} \}\ \tilde{\Psi} = \nu^2 \tilde{\Psi}, \qquad \nu = i\frac{2p}{\beta^2} \tag{40}$$

The potential term in (40) decays at $y \to -\infty$ and therefore for pure imaginary $p = -i\nu\beta^2/2$ (40) defines a scattering problem, the reflection scattering amplitude $S(\nu, E)$ being defined as usual as the coefficient in the $y \to -\infty$ asymptotic

$$\tilde{\Psi}(y) \to e^{i\nu y} + S(\nu, E)\, e^{-i\nu y} \qquad \text{as} \quad y \to -\infty \tag{41}$$

of the solution $\tilde{\Psi}(y)$ which decays at $y \to +\infty$. This coefficient is readily extracted from (26), (29) and (30),

$$S(\nu, E) = -\frac{\Gamma(1 + i\nu\beta^2)\, D(E, -1/2 + i\nu)}{\Gamma(1 - i\nu\beta^2)\, D(E, -1/2 - i\nu)} \left(\frac{\beta^2}{2}\right)^{-2i\nu\beta^2} \tag{42}$$

Using the relations (32) we have for (39)

$$J(V) = V - 2i\pi TE\, \partial_E \log S\left(\frac{V}{2\pi T}, E\right), \qquad E = -\frac{\kappa^2\pi^2}{\Gamma^2(1+\beta^2)}\, (\pi T)^{2\beta^2 - 2} \tag{43}$$

i.e., the nonlinear mobility (39) is expressed in terms of the scattering phase in the Schrödinger problem (40). Note that this expression allows one to prove the duality relation $\beta^2 \to 1/\beta^2$ for $J(V)$, first proposed in ref. 12, by simple change of the variables in (40).

In conclusion, let us remark that the main ingredient used in derivation of the functional equation (33) is the symmetries (21), (22) of the Schrödinger operator (12). Clearly, it is possible to modify this operator while preserving these symmetries. This suggests that the relation between the eigenvalues of **Q**-operators of CFT and spectral characteristics of Schrödinger operators may be generalizable to excited-states eigenvalues of $\mathbf{Q}_\pm$ and, more importantly, to **Q**-operators corresponding to massive integrable field theories (sine-Gordon model).

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NOTE ADDED

After this work was completed and posted in hep-th/9812247, some further interesting developments of the Doorey–Tateo relation appeared in the literature, see refs. 13–15. Other related recent works are ref. 16.

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